Fierz transformations

Fierz identities are often useful in quantum field theory calculations. They are connected to reordering of field operators in a contact four-particle interaction. The basic task is: given four complex fields $\psi_{1,2,3,4}$, which carry a (spacetime or internal) index, let us consider an interaction

$$(\overline{\psi}_1 A \psi_2)(\overline{\psi}_3 B \psi_4). \tag{1}$$

The indices of the spinors are suppressed. The same interaction can be expressed in a different way as $(\overline{\psi}_1 M \psi_4)(\overline{\psi}_3 N \psi_2)$. *How are the matrices M, N related to the matrices A, B?* The obvious answer is

$$A_{ij} B_{kl} = \pm M_{il} N_{kj}, \tag{2}$$

where the plus and minus signs apply to bosonic and fermionic field operators, respectively. However, it is usually inconvenient to use the matrix elements explicitly and the matrices A, B, M, N are often expressed in a suitable basis.

General Fierz identity

Denoting the vector space of the spinors as $\mathcal{R}$, let as assume that we know a basis in the matrix space $\overline{\mathcal{R}} \otimes \mathcal{R}$; we will call it Γ_a. The scalar product of basis matrices gives rise to a metric,[1]

$$\mathrm{Tr}(\Gamma_a \Gamma_b) = g_{ab},$$

which may be used to raise and lower indices by[2] $\Gamma^a = \sum_b g^{ab} \Gamma_b$, as usual. Every matrix M can be expanded in this basis as

$$M = \sum_a M^a \Gamma_a, \quad \text{where} \quad M^a = \mathrm{Tr}(M \Gamma^a).$$

This leads immediately to the completeness relation,

$$\sum_a (\Gamma_a)_{ij} (\Gamma^a)_{kl} = \delta_{il} \delta_{jk}, \tag{3}$$

which is the basis for the derivation of all Fierz identities.

The above introduced matrices A, B, M, N are accordingly written as

$$A_{ij} B_{kl} = \sum_{a,b} A^a B^b (\Gamma_a)_{ij} (\Gamma_b)_{kl}, \quad M_{ij} N_{kl} = \sum_{a,b} M^a N^b (\Gamma_a)_{ij} (\Gamma_b)_{kl}.$$

---

[1] In fact, for general complex matrices we should Hermitian conjugate one of the matrices in the trace to get a well-defined scalar product. Therefore, all our conclusions will hold without further assumptions in case the Γ_as are Hermitian. Otherwise, we need to suppose at least that g_{ab} is invertible. This is indeed the case in all applications.

[2] Here and in the following, all sums over indices will be indicated explicitly.

The task to find the relation (2) between them is therefore equivalent to finding the relation between $(\Gamma_a)_{ij}(\Gamma_b)_{kl}$ and $(\Gamma_a)_{il}(\Gamma_b)_{kj}$. This is in general given by a linear combination

$$(\Gamma_a)_{ij}(\Gamma_b)_{kl} = \sum_{c,d} C_{ab}{}^{cd}(\Gamma_c)_{il}(\Gamma_d)_{kj}. \tag{4}$$

Multiplying by $(\Gamma_e)_{li}(\Gamma_f)_{jk}$, we infer immediately

$$C_{abcd} = \mathrm{Tr}(\Gamma_a\Gamma_d\Gamma_b\Gamma_c). \tag{5}$$

Equations (4) and (5) represent the most general Fierz rearrangement formula. However, in practice one does not usually need to calculate the coefficients C_{abcd} for all combinations of indices. Thanks to symmetry there are typically just a few independent ones.

Symmetry constraints

The space of spinors $\mathcal{R}$ furnishes an irreducible representation of a symmetry group, under which the interaction Lagrangian is required to be invariant. This is most easily accomplished in two consecutive steps. First the product representation $\overline{\mathcal{R}} \otimes \mathcal{R}$ is decomposed into irreducible representations of the symmetry group using the set of Clebsch–Gordan coefficients. For two spinors ψ, χ in the same representation, the set of bilinears

$$\overline{\psi}\Gamma_a^{\mathcal{A}}\chi, \quad a = 1, \ldots, \dim\mathcal{A},$$

form a basis of the irreducible representation $\mathcal{A}$ that lies in the decomposition of $\overline{\mathcal{R}} \otimes \mathcal{R}$. For any selected representation $\mathcal{A}$ we can form a unique invariant of the symmetry group,[3] $(\overline{\psi}_1\Gamma_a^{\mathcal{A}}\psi_2)(\overline{\psi}_3\Gamma^{\mathcal{A}a}\psi_4)$.

We therefore need not rearrange the products of all pairs of two individual basis matrices, but rather the sum, $\sum_a \Gamma_a^{\mathcal{A}} \otimes \Gamma^{\mathcal{A}a}$, over all matrices in a given irreducible representation. The Fierz transformation analogous to the general formula (4) will then read

$$\sum_a (\Gamma_a^{\mathcal{A}})_{ij}(\Gamma^{\mathcal{A}a})_{kl} = \sum_{\mathcal{B}} C_{\mathcal{A}\mathcal{B}} \sum_b (\Gamma_b^{\mathcal{B}})_{il}(\Gamma^{\mathcal{B}b})_{kj}, \tag{6}$$

where the Fierz coefficients now depend only on the representations in question. Summing Eq. (4) over all $a = b$ in a given representation and realizing that by symmetry considerations, $C_a{}^a{}_c{}^d$ can only be nonzero for $c = d$, we find

$$\sum_a (\Gamma_a^{\mathcal{A}})_{ij}(\Gamma^{\mathcal{A}a})_{kl} = \sum_{\mathcal{B}} \sum_{a,b} C_a{}^{ab}{}_b(\Gamma_b^{\mathcal{B}})_{il}(\Gamma^{\mathcal{B}b})_{kj},$$

and from (5) then

$$C_{\mathcal{A}\mathcal{B}} = \sum_a C_a{}^{ab}{}_b = \sum_a \mathrm{Tr}(\Gamma_a^{\mathcal{A}}\Gamma_b^{\mathcal{B}}\Gamma^{\mathcal{A}a}\Gamma^{\mathcal{B}b}). \tag{7}$$

---

[3]The situation would become slightly more complicated in case the decomposition of $\overline{\mathcal{R}} \otimes \mathcal{R}$ contained more equivalent irreducible representations. For the sake of simplicity, we neglect this possibility here.

Properties of Fierz coefficients

[1] By multiplying Eq. (6) with $\Gamma^{\mathcal{C}}_{c,jk}$ and using the orthogonality condition in the form

$$\mathrm{Tr}(\Gamma^{\mathcal{A}}_a \Gamma^{\mathcal{B}}_b) = \delta^{\mathcal{AB}} g_{ab}, \tag{8}$$

we derive a very useful formula

$$\boxed{\sum_a \Gamma^{\mathcal{A}}_a \Gamma^{\mathcal{B}}_b \Gamma^{\mathcal{A}a} = C_{\mathcal{AB}} \Gamma^{\mathcal{B}}_b,} \tag{9}$$

which is often more convenient to evaluate the coefficients $C_{\mathcal{AB}}$ than the definition (7).

[2] Another distinguishing property of the Fierz transformation is that performing it twice, we get back to the original interaction, that is, the Fierz transformation is equal to its inverse. In terms of the matrix of coefficients $C_{\mathcal{AB}}$, this can be seen by applying Eq. (6) to itself,

$$\sum_a (\Gamma^{\mathcal{A}}_a)_{ij}(\Gamma^{\mathcal{A}a})_{kl} = \sum_{\mathcal{B}} C_{\mathcal{AB}} \sum_b (\Gamma^{\mathcal{B}}_b)_{il}(\Gamma^{\mathcal{B}b})_{kj} = \sum_{\mathcal{B}} C_{\mathcal{AB}} \sum_{\mathcal{C}} C_{\mathcal{BC}} \sum_c (\Gamma^{\mathcal{C}}_c)_{ij}(\Gamma^{\mathcal{C}c})_{kl},$$

$$\boxed{\sum_{\mathcal{B}} C_{\mathcal{AB}} C_{\mathcal{BC}} = \delta_{\mathcal{AC}}.} \tag{10}$$

[3] In practice the representation $\mathcal{R}$ often is a representation of a direct product of groups, corresponding to different quantum numbers such as spin, flavor, or color. We therefore need to know how to perform the Fierz transformation with respect to several indices. Let us denote the basis of matrices in the product representation $\mathcal{A} \otimes \mathcal{A}'$ as $\Gamma^{\mathcal{AA}'}_{aa'} \equiv \Gamma^{\mathcal{A}}_a \otimes \Gamma^{\mathcal{A}'}_{a'}$. Applying repeatedly Eq. (9), we obtain

$$\sum_{aa'} \Gamma^{\mathcal{AA}'}_{aa'} \Gamma^{\mathcal{BB}'}_{bb'} \Gamma^{\mathcal{AA}'aa'} = \sum_{aa'} (\Gamma^{\mathcal{A}}_a \Gamma^{\mathcal{B}}_b \Gamma^{\mathcal{A}a}) \otimes (\Gamma^{\mathcal{A}'}_{a'} \Gamma^{\mathcal{B}'}_{b'} \Gamma^{\mathcal{A}'a'}) =$$

$$= \left(\sum_a \Gamma^{\mathcal{A}}_a \Gamma^{\mathcal{B}}_b \Gamma^{\mathcal{A}a}\right) \otimes \left(\sum_{a'} \Gamma^{\mathcal{A}'}_{a'} \Gamma^{\mathcal{B}'}_{b'} \Gamma^{\mathcal{A}'a'}\right) = C_{\mathcal{AB}} C_{\mathcal{A}'\mathcal{B}'} \Gamma^{\mathcal{B}}_b \otimes \Gamma^{\mathcal{B}'}_{b'} = C_{\mathcal{AB}} C_{\mathcal{A}'\mathcal{B}'} \Gamma^{\mathcal{BB}'}_{bb'},$$

and consequently

$$\boxed{C_{\mathcal{AA}',\mathcal{BB}'} = C_{\mathcal{AB}} C_{\mathcal{A}'\mathcal{B}'}.}$$

This is a great simplification which tells us that the Fierz transformation can be performed on each index separately.

[4] Summing Eq. (7) over b, we get a result invariant under the exchange $\mathcal{A} \leftrightarrow \mathcal{B}$, which implies a nice reciprocity relation

$$\boxed{C_{\mathcal{AB}} \dim \mathcal{B} = C_{\mathcal{BA}} \dim \mathcal{A}.} \tag{11}$$

[5] The representation $\overline{\mathcal{R}} \otimes \mathcal{R}$ always contains the unit representation, $\mathcal{I}$, and the corresponding Clebsch–Gordan coefficients are conveniently defined by the unit matrix, $\Gamma_{\mathcal{I}} = \mathbb{1}$. Assuming

that all other basis matrices are chosen orthogonal to $\mathbb{1}$, i.e. traceless, we find $\Gamma^{\mathcal{I}} = \mathbb{1}/\operatorname{Tr}\mathbb{1} = \mathbb{1}/\dim\mathcal{R}$. Substituting $\mathcal{A} = \mathcal{I}$ in (9) then leads to

$$\boxed{C_{\mathcal{I}\mathcal{A}} = \frac{1}{\dim\mathcal{R}}, \quad C_{\mathcal{A}\mathcal{I}} = \frac{\dim\mathcal{A}}{\dim\mathcal{R}},} \tag{12}$$

the second relation following immediately from (11).

Examples

[1] $\mathfrak{su}(N)$ algebra

In this case, $\mathcal{R}$ will be the fundamental representation of $\mathfrak{su}(N)$ with the generators T_a normalized by $\operatorname{Tr}(T_a T_b) = \xi\delta_{ab}$. The representation $\overline{\mathcal{R}} \otimes \mathcal{R}$ decomposes into the sum of the adjoint representation $\mathcal{T}$, with $\Gamma_a^{\mathcal{T}} = T_a$, and the unit representation $\mathcal{I}$, with $\Gamma_{\mathcal{I}} = \mathbb{1}$. Note that the normalization of $\mathbb{1}$, given by $g_{\mathcal{I}\mathcal{I}} = N$, in general differs from the normalization of T_a! With the help of equations (7) and (11) we easily deduce

$$C_{\mathcal{I}\mathcal{I}} = \frac{1}{N}, \quad C_{\mathcal{I}\mathcal{T}} = \frac{1}{N}, \quad C_{\mathcal{T}\mathcal{I}} = \frac{N^2-1}{N}.$$

The same result is an immediate consequence of Eq. (12). The last missing Fierz coefficient, $C_{\mathcal{T}\mathcal{T}}$, follows most easily from (10): $C_{\mathcal{T}\mathcal{T}} = -1/N$. We can now summarize the Fierz transformations for $\mathfrak{su}(N)$ in the conventional way, without using raised indices,

$$(\mathbb{1})_{ij}(\mathbb{1})_{kl} = \frac{1}{N}(\mathbb{1})_{il}(\mathbb{1})_{kj} + \frac{1}{\xi}\sum_a (T_a)_{il}(T_a)_{kj},$$

$$\sum_a (T_a)_{ij}(T_a)_{kl} = \xi\frac{N^2-1}{N^2}(\mathbb{1})_{il}(\mathbb{1})_{kj} - \frac{1}{N}\sum_a (T_a)_{il}(T_a)_{kj}. \tag{13}$$

From here, or directly from Eq. (9), we then immediately obtain another useful identity,

$$\sum_a T_a T_b T_a = -\frac{\xi}{N}T_b.$$

Equations (13) were derived for the fundamental representation of $\mathfrak{su}(N)$. Even for higher representations our general formulas can still yield some (restricted) information. Let us therefore consider the $\mathfrak{su}(N)$ generators $T_a^{\mathcal{R}}$ in an arbitrary representation $\mathcal{R}$. From the group-theoretic point of view, they define the Clebsch–Gordan coefficients for the adjoint representation $\mathcal{T}$ in the decomposition of the direct product $\overline{\mathcal{R}} \otimes \mathcal{R}$. Their norm is usually denoted as $C(\mathcal{R})$, that is, $\operatorname{Tr}(T_a^{\mathcal{R}} T_b^{\mathcal{R}}) = C(\mathcal{R})\delta_{ab}$. Eq. (12) gives $C_{\mathcal{T}\mathcal{I}} = (N^2-1)/\dim\mathcal{R}$. Applying Eq. (9) to $\mathcal{B} = \mathcal{I}$ then results in the conventional quadratic Casimir invariant, expressed as

$$\sum_a T_a^{\mathcal{R}} T_a^{\mathcal{R}} = C_2(\mathcal{R})\mathbb{1}^{\mathcal{R}}, \quad \text{where} \quad C_2(\mathcal{R}) = C(\mathcal{R})\frac{N^2-1}{\dim\mathcal{R}}.$$

[2] Dirac algebra

Proceeding in the same manner one can derive the Fierz identities for an arbitrary matrix algebra. Here we quote some results for the algebra of Dirac matrices, without a detailed derivation, which is straightforward, though a bit tedious. The standard Lorentz-covariant basis of 4×4 matrices is created from the scalar, vector, tensor, axial-vector, and pseudoscalar combinations of Dirac γ-matrices, $\{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \mathrm{i}\gamma_5\}$, where we use the conventions $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$ and $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. In many physical applications, the Lorentz symmetry is nevertheless broken by the presence of a dense medium down to mere rotation symmetry. One therefore needs to work with the basis of rotation-covariant matrices, that is, $\{\mathbb{1}, \gamma_0, \gamma^a, \sigma^{a0}, \sigma^{ab}, \gamma_0\gamma_5, \gamma^a\gamma_5, \mathrm{i}\gamma_5\}$. The indices a, b now run from one to three. The resulting Fierz identities for rotation-invariant bilinears are summarized in Fig. 1. The first matrix gives the Fierz coefficients defined by (6), while the second matrix correspond to the Fierz transformation to the particle–particle channel, discussed below.

Fierz transformation in the particle–particle channel

Of the four fields in the expression (1), two transform in the representation $\mathcal{R}$ and two in its complex conjugate $\overline{\mathcal{R}}$. In the above two-step construction we created invariants of a given symmetry from the product representations $(\overline{\mathcal{R}} \otimes \mathcal{R})$ and $(\overline{\mathcal{R}} \otimes \mathcal{R})$. The two possible ways to compose $\overline{\mathcal{R}} \otimes \mathcal{R}$ gave rise to the Fierz rearrangement identity (6).

However, we can also construct an invariant by first composing $(\mathcal{R} \otimes \mathcal{R})$ and $(\overline{\mathcal{R}} \otimes \overline{\mathcal{R}})$, and then combining those. We will for simplicity refer to this rearrangement as the particle–particle one, for obvious reasons. The product representation $\mathcal{R} \otimes \mathcal{R}$ naturally has a different basis than $\Gamma_a^{\mathcal{A}}$; we will denote the corresponding matrices as $\Xi_a^{\mathcal{A}}$. The new Fierz transformation will therefore be defined analogously to (6) as

$$\sum_a (\Gamma_a^{\mathcal{A}})_{ij}(\Gamma^{\mathcal{A}a})_{kl} = \sum_{\mathcal{B}} D_{\mathcal{A}\mathcal{B}} \sum_b (\Xi_b^{\mathcal{B}})_{ik}(\overline{\Xi}^{\mathcal{B}b})_{lj}. \tag{14}$$

The matrices $\overline{\Xi}$, forming the basis of $\overline{\mathcal{R}} \otimes \overline{\mathcal{R}}$, are most generally defined by $(\overline{\psi}\Xi\overline{\chi}^T)^\dagger = \chi^T\overline{\Xi}\psi$. Note that we will not derive the Fierz transformation to the particle–particle channel in the most general case, analogously to Eq. (4). The generalization is obvious, but of little practical utility.

The basis matrices $\Xi_a^{\mathcal{A}}$ will be assumed to be normalized similarly to (8),

$$\mathrm{Tr}(\Xi_a^{\mathcal{A}}\overline{\Xi}_b^{\mathcal{B}}) = \delta^{AB}h_{ab},$$

and the metric h_{ab} will again be used to raise and lower indices. Analogously to Eqs. (7) and (9) we derive the identities

$$D_{\mathcal{A}\mathcal{B}} = \sum_a \mathrm{Tr}\big[\Gamma_a^{\mathcal{A}}\Xi_b^{\mathcal{B}}(\Gamma^{\mathcal{A}a})^T\overline{\Xi}^{\mathcal{B}b}\big], \tag{15}$$

$$\sum_a \Gamma_a^{\mathcal{A}} \Xi_b^{\mathcal{B}} (\Gamma^{\mathcal{A}a})^T = D_{\mathcal{AB}} \Xi_b^{\mathcal{B}}. \tag{16}$$

The coefficients $C_{\mathcal{AB}}$ and $D_{\mathcal{AB}}$ arise from rearrangement of the same matrices. It is therefore not surprising that they are related to each other. To see this, let us apply the identity (6) to the left-hand side of Eq. (16),

$$\sum_a \left[\Gamma_a^{\mathcal{A}} \Xi_b^{\mathcal{B}} (\Gamma^{\mathcal{A}a})^T \right]_{ij} = \sum_{a,k,l} (\Gamma_a^{\mathcal{A}})_{ik} (\Gamma^{\mathcal{A}a})_{jl} (\Xi_b^{\mathcal{B}})_{kl} = \sum_{\mathcal{C}} C_{\mathcal{AC}} \sum_{c,k,l} (\Gamma_c^{\mathcal{C}})_{il} (\Gamma^{\mathcal{C}c})_{jk} (\Xi_b^{\mathcal{B}})_{kl} =$$

$$= \sum_{\mathcal{C}} C_{\mathcal{AC}} \sum_c \left[\Gamma_c^{\mathcal{C}} (\Xi_b^{\mathcal{B}})^T (\Gamma^{\mathcal{C}c})^T \right]_{ij}.$$

Being a basis of the representation $\mathcal{R} \otimes \mathcal{R}$, the matrices $\Xi_b^{\mathcal{B}}$ are always either symmetric or antisymmetric, that is, $(\Xi_b^{\mathcal{B}})^T = \eta_{\mathcal{B}} \Xi_b^{\mathcal{B}}$, where the sign $\eta_{\mathcal{B}} = \pm$ depends only on the irreducible representation $\mathcal{B}$. Applying Eq. (16) to the first and last expression above, we obtain the relation

$$D_{\mathcal{AB}} = \eta_{\mathcal{B}} \sum_{\mathcal{C}} C_{\mathcal{AC}} D_{\mathcal{CB}}. \tag{17}$$

Setting $\mathcal{A}$ to the unit representation $\mathcal{I}$, we obtain a useful special case

$$D_{\mathcal{IA}} = \frac{1}{\dim \mathcal{R}}, \quad 1 = \eta_{\mathcal{B}} \sum_{\mathcal{A}} D_{\mathcal{AB}}. \tag{18}$$

The second identity follows from (12). Although the relations (17) and (18) only constrain the coefficients $D_{\mathcal{AB}}$, in some special cases they may actually be sufficient to determine $D_{\mathcal{AB}}$ completely.

$\mathfrak{su}(N)$ algebra

The product $\mathcal{R} \otimes \mathcal{R}$ of two fundamental representations of $\mathfrak{su}(N)$ decomposes into two irreducible representations, the symmetric and antisymmetric tensors, $\mathcal{S}$ and $\mathcal{A}$. We will denote the corresponding basis matrices respectively as S_a and A_a. Note that A_a are simply the antisymmetric T_a matrices, while S_a are the symmetric T_a matrices together with the unit matrix, normalized to have the same norm ξ, that is, $\mathbb{1}\sqrt{\xi/N}$. Now the first identity in (18) immediately yields $D_{\mathcal{IS}} = D_{\mathcal{IA}} = 1/N$. Given that $\eta_{\mathcal{S}} = 1$ and $\eta_{\mathcal{A}} = -1$, the second identity then implies

$$D_{\mathcal{TS}} = \frac{N-1}{N}, \quad D_{\mathcal{TA}} = -\frac{N+1}{N}.$$

We can therefore summarize the Fierz identities analogous to (13),

$$(\mathbb{1})_{ij}(\mathbb{1})_{kl} = \frac{1}{\xi} \sum_a (S_a)_{ik}(S_a)_{lj} + \frac{1}{\xi} \sum_a (A_a)_{ik}(A_a)_{lj},$$

$$\sum_a (T_a)_{ij}(T_a)_{kl} = \frac{N-1}{N} \sum_a (S_a)_{ik}(S_a)_{lj} - \frac{N+1}{N} \sum_a (A_a)_{ik}(A_a)_{lj}.$$

Particle–particle channel for (pseudo)real representations

Often, the representation $\mathcal{R}$ is (pseudo)real, i.e., it is equivalent to its complex conjugate. Then the matrices $\Xi_a^{\mathcal{A}}$ can be directly related to $\Gamma_a^{\mathcal{A}}$ and the coefficients $D_{\mathcal{AB}}$ to $C_{\mathcal{AB}}$. Let us assume that the equivalence is provided by the unitary matrix Q, that is, ψ transforms in the same way as $Q\overline{\psi}^T$. Then the natural choice for the matrices $\Xi_a^{\mathcal{A}}$ is $\Xi_a^{\mathcal{A}} = \Gamma_a^{\mathcal{A}}Q$, and accordingly $\overline{\Xi}_a^{\mathcal{A}} = \overline{Q}\Gamma_a^{\mathcal{A}}$. (We tacitly assume $\overline{\Gamma}_a^{\mathcal{A}} = \Gamma_a^{\mathcal{A}}$, which can always be ensured by a suitable definition.) (Anti)symmetry of the matrices $\Xi_a^{\mathcal{A}}$, encoded in the sign $\eta_{\mathcal{A}}$, then implies

$$(\Gamma_a^{\mathcal{A}})^T = \eta_{\mathcal{A}}(Q^T)^{-1}\Gamma_a^{\mathcal{A}}Q.$$

According to the general theory of Lie algebra representations, the matrix Q itself is either symmetric or antisymmetric. In fact, it is symmetric if the representation $\mathcal{R}$ is real, and antisymmetric if $\mathcal{R}$ is pseudoreal. We shall therefore write generally $Q^T = \eta_Q Q$. Then

$$(\Gamma_a^{\mathcal{A}})^T = \eta_Q \eta_{\mathcal{A}} Q^{-1}\Gamma_a^{\mathcal{A}}Q.$$

The coefficients $D_{\mathcal{AB}}$ now follow from Eq. (15),

$$D_{\mathcal{AB}} = \sum_a \text{Tr}\left[\Gamma_a^{\mathcal{A}}\Xi_b^{\mathcal{B}}(\Gamma^{\mathcal{A}a})^T\overline{\Xi}^{\mathcal{B}b}\right] = \eta_{\mathcal{A}}\eta_Q \sum_a \text{Tr}\left[\Gamma_a^{\mathcal{A}}\Gamma_b^{\mathcal{B}}QQ^{-1}\Gamma^{\mathcal{A}a}Q\overline{Q}\Gamma^{\mathcal{B}b}\right].$$

This already looks almost like the expression (7) for $C_{\mathcal{AB}}$, were it not for the matrix product $Q\overline{Q}$. However, this must actually be proportional to the unit matrix. To see this, note that since $\psi' = Q\overline{\psi}^T$ transforms in the same way as ψ, $\overline{\psi}'\psi'$ must be a group invariant. An easy manipulation shows that $\overline{\psi}'\psi' = \psi^T\overline{Q}Q\overline{\psi}^T = \pm\overline{\psi}Q\overline{Q}\psi$, the $\pm$ sign referring to bosons or fermions. The invariance of the last expression requires that $Q\overline{Q}$ commutes with all matrices of the representation $\mathcal{R}$, hence by Schur's lemma must be proportional to the unit matrix as long as the representation $\mathcal{R}$ is irreducible. Let us write $Q\overline{Q} = \overline{\eta}\mathbb{1}$, where $\overline{\eta}$ is a complex unit since Q is unitary. We then immediately obtain the simple final result

$$\boxed{D_{\mathcal{AB}} = \eta_{\mathcal{A}}\eta_Q\overline{\eta}C_{\mathcal{AB}}.} \tag{19}$$

Specifically for the algebra of Dirac γ-matrices, Q is the charge conjugation matrix. In the standard Dirac representation we find $\eta_Q = \overline{\eta} = -1$, the coefficients $D_{\mathcal{AB}}$ are thus related to $C_{\mathcal{AB}}$ simply by the sign, defining the (anti)symmetry of the representation $\mathcal{A}$. The values of all Fierz coefficients are summarized in Fig. 1.

$$
\begin{pmatrix}
\delta_{ij}\delta_{kl} \\
(\gamma_0)_{ij}(\gamma_0)_{kl} \\
(\gamma^a)_{ij}(\gamma_a)_{kl} \\
(\sigma^{a0})_{ij}(\sigma_{a0})_{kl} \\
\frac{1}{2}(\sigma^{ab})_{ij}(\sigma_{ab})_{kl} \\
(\gamma_0\gamma_5)_{ij}(\gamma_0\gamma_5)_{kl} \\
(\gamma^a\gamma_5)_{ij}(\gamma_a\gamma_5)_{kl} \\
(i\gamma_5)_{ij}(i\gamma_5)_{kl}
\end{pmatrix}
=
\begin{pmatrix}
1/4 & 1/4 & 1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 \\
1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & 1/4 \\
3/4 & -3/4 & -1/4 & 1/4 & -1/4 & -3/4 & -1/4 & 3/4 \\
3/4 & -3/4 & 1/4 & -1/4 & -1/4 & 3/4 & -1/4 & -3/4 \\
3/4 & 3/4 & -1/4 & -1/4 & -1/4 & -3/4 & 1/4 & -3/4 \\
-1/4 & 1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 & -1/4 \\
-3/4 & -3/4 & -1/4 & -1/4 & 1/4 & -3/4 & -1/4 & -3/4 \\
-1/4 & 1/4 & 1/4 & -1/4 & -1/4 & -1/4 & -1/4 & 1/4
\end{pmatrix}
\begin{pmatrix}
\delta_{il}\delta_{kj} \\
(\gamma_0)_{il}(\gamma_0)_{kj} \\
(\gamma^a)_{il}(\gamma_a)_{kj} \\
(\sigma^{a0})_{il}(\sigma_{a0})_{kj} \\
\frac{1}{2}(\sigma^{ab})_{il}(\sigma_{ab})_{kj} \\
(\gamma_0\gamma_5)_{il}(\gamma_0\gamma_5)_{kj} \\
(\gamma^a\gamma_5)_{il}(\gamma_a\gamma_5)_{kj} \\
(i\gamma_5)_{il}(i\gamma_5)_{kj}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\delta_{ij}\delta_{kl} \\
(\gamma_0)_{ij}(\gamma_0)_{kl} \\
(\gamma^a)_{ij}(\gamma_a)_{kl} \\
(\sigma^{a0})_{ij}(\sigma_{a0})_{kl} \\
\frac{1}{2}(\sigma^{ab})_{ij}(\sigma_{ab})_{kl} \\
(\gamma_0\gamma_5)_{ij}(\gamma_0\gamma_5)_{kl} \\
(\gamma^a\gamma_5)_{ij}(\gamma_a\gamma_5)_{kl} \\
(i\gamma_5)_{ij}(i\gamma_5)_{kl}
\end{pmatrix}
=
\begin{pmatrix}
-1/4 & -1/4 & -1/4 & -1/4 & -1/4 & 1/4 & 1/4 & 1/4 \\
1/4 & 1/4 & -1/4 & -1/4 & 1/4 & 1/4 & -1/4 & 1/4 \\
3/4 & -3/4 & -1/4 & 1/4 & -1/4 & -3/4 & -1/4 & 3/4 \\
3/4 & -3/4 & 1/4 & -1/4 & -1/4 & 3/4 & -1/4 & -3/4 \\
3/4 & 3/4 & -1/4 & -1/4 & -1/4 & -3/4 & 1/4 & -3/4 \\
1/4 & -1/4 & 1/4 & -1/4 & 1/4 & -1/4 & 1/4 & 1/4 \\
3/4 & 3/4 & 1/4 & 1/4 & -1/4 & 3/4 & 1/4 & 3/4 \\
1/4 & -1/4 & -1/4 & 1/4 & 1/4 & 1/4 & 1/4 & -1/4
\end{pmatrix}
\begin{pmatrix}
C_{ik}C_{lj} \\
(\gamma_0 C)_{ik}(C\gamma_0)_{lj} \\
(\gamma^a C)_{ik}(C\gamma_a)_{lj} \\
(\sigma^{a0}C)_{ik}(C\sigma_{a0})_{lj} \\
\frac{1}{2}(\sigma^{ab}C)_{ik}(C\sigma_{ab})_{lj} \\
(\gamma_0\gamma_5 C)_{ik}(C\gamma_0\gamma_5)_{lj} \\
(\gamma^a\gamma_5 C)_{ik}(C\gamma_a\gamma_5)_{lj} \\
(i\gamma_5 C)_{ik}(iC\gamma_5)_{lj}
\end{pmatrix}
$$

Figure 1: Fierz transformation of rotation-invariant Dirac bilinears into the particle–antiparticle and particle–particle channels.